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A Model of Intangible Capital

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Abstract

The researchers propose a model that starts from the premise that intangible capital needs to be stored on some medium – software, patents, essential employees – before it can be utilized in production. Storage implies that intangible capital may be partially non-rival within the firm, leading to scale economies. However, storage can also compromise the ability of the firm to fully appropriate the returns generated by intangibles. The authors explore the implications of these two mechanisms for firm scale, scope, and investment decisions, and they outline their connection to recent macroeconomic and financial trends in the U.S.

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1 Introduction

Over the last three decades, intangibles have become an increasingly important part the productive capital stock of US corporations (Eisfeldt and Papanikolaou, 2014; Crouzet and Eberly, 2018). Important and familiar examples of intangible assets include patents, software and databases, video and audio material, franchise agreements, consumer lists, organization capital, and brands.

Intangibles are often defined in the negative: that is, they are productive capital that lacks a physical presence. In keeping with this approach, existing research generally treats intangibles as another kind of capital within the standard neo-classical investment framework. Intangibles may require different time to build than physical capital; they may be subject to different fixed costs; they may depreciate more slowly; they may have a different relative price; and they may be more difficult to measure. But these differences are generally a matter of degree, not a matter of kind.[1]

In this paper, we propose to focus instead on the properties that affirmatively characterize intangibles. At an abstract level, intangibles consist of information. In order to be put to productive use, this information needs to be stored. The storage medium can be a document (say, for a design or patent), a piece of software (say, for an algorithm), or a person (say, for an organizational innovation). We argue that the need-to-store has two key economic implications. First, need-to-store implies some degree of *non-rivalry* in use within the firm. That is, to the extent that the storage medium can easily be replicated within the firm, the same intangible can be used *simultaneously* stored and used by the firm across *multiple* locations — for instance, the same brand can be used by a firm across multiple retail locations. Second, need-to-store can also imply limits to *excludability*. To the extent that the intangible is easy to copy, it can also be appropriated by outsiders — for instance, by copying the software that the intangible is embedded in. This potential for external appropriation makes it more difficult to establish intangibles as assets with clearly identified control and cash flow rights.

Our main contribution is to propose a simple model that captures these two core economic properties of intangibles — non-rivalry within the firm, and limits to excludability. The model has a number of novel theoretical implications. Non-rivalry implies that intangibles are ”scalable”, in a specific sense: the stock of intangibles of the firm, and the firm's span or scope, are complements. However, imperfect excludability can also limit the incentive for entrepreneurs, managers, or key personnel to create and develop intangibles, potentially leading to inefficiently low investment.

We then explore the implications of the model for four recent macroeconomic and financial trends: the decline in measured aggregate productivity; the rise in the labor income share; the growing divergence between Tobin's Q and investment; and the rise in rents. We discuss how both

[1] Examples of this approach include, for instance, Hall (2001), McGrattan and Prescott (2010), as well as our own work; for instance, Crouzet and Eberly (2021).

non-rivalry and limits to excludability can contribute to understanding these trends, contrasting in particular our results with traditional approaches to measuring intangibles.

The need to store intangibles arises from the fact that they primarily consist of information, which cannot be widely communicated or used until it is codified. The storage medium depends on the intangible: a logo can be stored as a drawing; a new drug can be stored as a patent; a managerial innovation can be stored in a set of written instructions, or by training personnel, or both. Though the storage medium and the intangible are often conflated, the two are not the same. An algorithm for managing relations between customers and product managers will be stored in a particular piece of software, but it is the algorithm itself, not the software, which creates economic value.

Given a storage medium, *non-rivalry* in use can arise when the storage medium can be copied and replicated at relatively low cost and with good accuracy within the firm. The firm can then, effectively, use the same asset in production simultaneously across several locations, production lines, and so on. For instance, J. Edgar Thomson, the President of the Pennsylvania Railroad (PPR) in the 1850s and 1860s, implemented identical and innovative organizational structure (the staff-and-line system) across the different roads that made up the PPR (Chandler, 1993). By contrast, physical assets are, by definition, rival in use: a particular truck cannot produce transportation services across different routes at the same time; the same mill cannot produce steel pipes in different locations at the same time. The degree of non-rivalry of intangibles within the firm can however depend on the underlying storage technology for the intangible. Intangibles that are primarily stored in key employees — such as organizational systems — may not be fully non-rival within the firm, because implementing widely may require reallocating scare managerial talent from other uses. Our model captures this notion of partial non-rivalry in a parsimonious way, allowing for full rivalry (as in the case of physical capital) and full non-rivalry as special cases.

Even once the intangible is stored in a particular medium, it can be difficult to assign control and cash flow rights to the surplus that it creates. We refer to this property as *limited excludability*. Without some degree of excludability, intangible capital cannot become an intangible *asset*. An extreme example of intangibles with no excludability are public capital goods, like open-source software. These inputs non-rival in use within a given firm, but it is not possible to exclude other firms from using them. At the other end of the spectrum, property and control rights are generally easier to assign to physical assets. Trucks must be titled, and the title identifies the owner. Ownership of a steel mill, while it might be shared, is formalized through contracts, and disputes regarding control generally have legal remedies. For intangibles, by contrast, property rights may not be clearly defined or easy to assign, so that the private returns to the creator of the intangible may not be equal to the total surplus that the intangible creates. For instance, key employees that have been trained to follow a particular managerial system can be hired away from the firm; a brand might be imitated; a firm may need to share specific details on production processes with another firm with whom they would like to engage in a joint venture. Our model also captures limits to excludability in a relatively parsimonious manner, with full excludability (as in the case of physical assets with well-defined ownership rights) and no excludability (as in the case of public capital goods) as special limits.

The analytical separation between non-rivalry and limits to excludability in our model captures the view that technology plays an important role in determining non-rivalry within the firm, while the property rights environment has a stronger influence of limits to excludability. However, we acknowledge that institutions and technology mutually influence one another. Institutions can influence the incentive for an entrepreneur to store an intangible in a particular medium. For instance, it may be easy and convenient to store audio or video material digitally, but if ownership rights over digital copies are poorly enforced, content creators might prefer to use other storage media. Conversely, reliable storage media, like detailed and easy to access patents, can make the enforcement of property rights easier. While we highlight how limit to excludability might influence an entrepreneur's choice between storage technologies, our model does not directly speak to the potential for both to be simultaneously determined.

The rest of the paper is organized as follows. Section 2 describes the simple model of production and investment with intangibles, and discusses its key theoretical predictions, contrasting them with existing work on intangibles. Section 3 derives the implications of the model for four macroeconomic and financial that have attracted attention in the US over the last few years. Section 4 concludes.

2 Production with intangible capital

In this section, we present a simple model which captures the key properties of intangible capital discussed in the introduction: need-to-store; non-rivalry within firm; and limited excludability. We discuss its main mechanisms and compare it to existing approaches. The model is also informally discussed in our companion paper, Crouzet, Eberly, Eisfeldt, and Papanikolaou (2022).

2.1 A simple model

We focus on a single firm which operates for one period. The firm is managed by an entrepreneur, who makes operating and investment choices to maximize the terminal value of the profits she receives. The model includes two types of capital, physical and intangible. Both types of capital can be deployed across multiple production streams, which could be different product lines, physical locations, or market segments. Together, these streams determine the span, x, of the firm.

To highlight the role of intangibles, we minimize the role of physical capital: it can be rented at

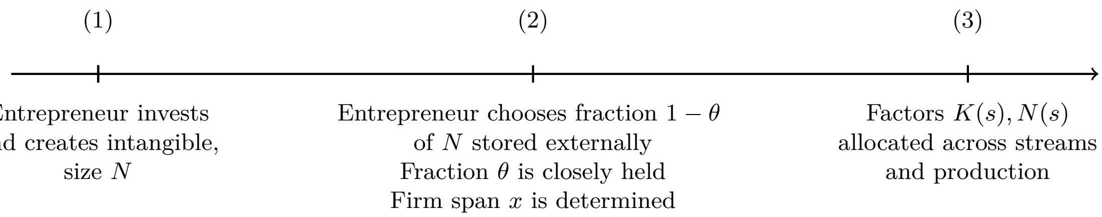

Figure 1: Timeline of the model.

a constant user cost. By contrast, intangibles in the model need to be stored but may be non-rival in use. Moreover, limited excludability will limit the scope of deployment of intangible capital. The entrepreneur contributes to the creation of the intangible capital, and owns a part or all of it.

For ease of exposition, we split the model into three stages. Figure 1 sketches the timeline of decisions in the model. We next discuss the choice of physical capital, firm span, and intangibles in the reverse order in which they occur, starting with the production stage.

Stage 3: Production

In the production stage, the firm's input N of intangible capital and its span of production x are taken as given. Profit maximization is described by:

$$V(N,x) = \max_{\{N(s),K(s)\}_{s\in[0,x]},K} \int_0^x F(K(s),N(s))ds - RK$$

$$\text{s.t.} \quad \int_0^x K(s)\,ds \leq K \tag{1}$$

$$\left(\int_0^x N(s)^{\frac{1}{1-\rho}}\,ds\right)^{1-\rho} \leq N \tag{2}$$

The firm thus chooses the total amount of physical capital K, and the allocation of physical and intangible capital to each stream of production, $\{K(s), N(s)\}$. Each $s \in [0, x]$ indexes a different production stream. Within each stream, production uses two inputs, intangible capital, $N(s)$ and physical capital $K(s)$, and has constant returns to scale:

$$\forall s \in [0,x], \quad Y(s) = F(K(s),N(s)) = N(s)^{1-\zeta}K(s)^{\zeta}, \tag{3}$$

where $\zeta \in \,]0,1[$ is the elasticity of output with respect to physical capital. A production stream could represent an establishment, a product, a market segment, a geography, so long as production satisfies constant returns within that stream.

The difference between the two constraints (1) and (2) illustrates the first fundamental property of intangible capital: because intangibles are non-rival in use, they are scalable in production.

In particular, the constraint limiting the deployment of physical capital K takes a familiar form, where the same unit of physical capital (say a machine) cannot be simultaneously used in multiple streams. The firm first decides on the total stock of physical capital K it wants to employ, and then rents it at the constant and exogenous price R on input markets. It then allocates that amount K across different production streams indexed by s, subject to the constraint (1).

Similarly, the firm's input of intangible capital N is allocated across streams subject to the constraint (2). Unlike the standard adding up constraint (1), however, the resource constraint for intangibles (2) is non-linear, capturing the idea that the same intangible can be used simultaneously in different streams.[2] The parameter ρ captures the degree of non-rivarly of the intangible input in production within the firm and across production streams.

Assumption 1 (non-rivalry within the firm)**.** $0 \leq \rho \leq 1$.

To see why ρ captures non-rivalry within the firm, consider two extreme cases. When $\rho = 0$, then there is no difference between physical and intangible capital: the two constraints (2) and (1) are identical and both types of capital are rival within the firm. By contrast, when ρ approaches 1, constraint (2) now becomes:

$$\lim_{\rho \to 1} \left(\int_0^x N(s)^{\frac{1}{1-\rho}} ds \right)^{1-\rho} = \max_{s \in [0,x]} N(s) \leq N. \tag{4}$$

In this case, the same intangible can be used in every production stream—that is, N becomes completely non-rival, since the firm can now set $N(s) = N$ for all streams.

More generally, the marginal rate of technical substitution of intangibles across any two production streams $N(s)$ and $N(s')$ is:

$$\nu(N(s), N(s'); \rho) = \left(\frac{N(s')}{N(s)} \right)^{\frac{\rho}{1-\rho}}.$$

This marginal rate of substitution is equal to 1 when $\rho = 0$, so that increasing $N(s')$ by a marginal unit requires reducing $N(s)$ by a marginal unit. When $\rho \to 1$, on the other hand, the marginal rate of substitution converges to:

$$\lim_{\rho \to 1} \nu(N(s), N(s'); \rho) = \begin{cases} +\infty & \text{if} \quad N(s') > N(s) \\ 0 & \text{if} \quad N(s') < N(s) \end{cases}$$

In this case, so long as $N(s') < N(s)$, increasing $N(s')$ by a marginal unit is costless: it does not require reducing $N(s)$ at all.

[2] Non-rivalry goes beyond assuming that the intangible input is not specific to a particular production stream. Rather, it implies that using the input across different streams does not incur an additional cost.

Our model allows for partial non-rivalry: $0 < \rho < 1$. In this case, increasing $N(s')$ by a small amount does require reducing $N(s)$, but less than one-for-one. In other words, increasing the intangible input in one production stream is not entirely costless, but it does not necessarily reduce its availability for other production streams, either.

Partial non-rivalry ($0 < \rho < 1$) captures intermediate situations, in which there are imperfections in the ability of the firm to replicate the intangible. For instance, in order to implement a new inventory management system at a particular production facility, a company may need to re-allocate existing managers or key employees to the facility. This re-allocation may negatively impact inventory management at other facilities, effectively reducing the intangible input (the inventory management system) at those facilities. Alternatively, a firm may share a customer list across several of its branches for promotion purposes, but only up to a point; an individual branch competing against another for the same customer would again reduce the effect of the intangible input (the customer list) on each individual branch's revenue.

Given our assumption that the marginal revenue product of all streams s is the same, the optimal allocation of capital across streams is symmetric:

$$\begin{aligned} \forall s \in [0, x]\,, \quad N(s) &= x^{-(1-\rho)}\, N, \\ K(s) &= \left(\frac{A}{1-\zeta}\right)^{\frac{1}{\zeta}} N(s), \qquad \text{where} \quad A \equiv (1-\zeta)\left(\frac{\zeta}{R}\right)^{\frac{\zeta}{1-\zeta}} \end{aligned}$$

As a result, the total demand for physical capital K and the value of the firm are equal to:

$$K(N,x) = \left(\frac{A}{1-\zeta}\right)^{\frac{1}{\zeta}} N\, x^{\rho}, \tag{5}$$

and

$$V(N,x) = A\, N\, x^{\rho}. \tag{6}$$

Examining (6), we can immediately see that the quantity of the intangible N and the scope of implementation x are complements, which can allow for increasing returns to scale if the firm can increase both N and x. We next introduce a trade-off between these two choices.

Stage 2: Storage choice and firm span

We next discuss the choice of scale, or span, x, taking, the initial endowment of intangible capital N, as given. Our key assumption is that in order to increase the span of the firm, the entrepreneur may need to give up some of the firm's endowment in intangibles.

We model this choice as follows. Let $0 \leq N_e \leq N$ denote the portion of intangible capital that the entrepreneur retains, and let:

$$\theta \equiv \frac{N_e}{N}.$$

Since V is linear in N, the total value accruing to the entrepreneur is:

$$V_e(N, x, \theta) = \theta V(N, x).$$

We make the following assumption about the relationship between firm span x and ownership θ.

Assumption 2 (Limited appropriability)**.** *The span of the firm is related to the share of intangibles retained by the entrepreneur through:*

$$x = f(\theta, \delta) = \begin{cases} -\frac{1}{\delta} \log\left(\frac{\theta}{\delta}\right) & \textit{if} \quad \theta \in [0, \delta) \\ 0 & \textit{if} \quad \theta \in [\delta, 1] \end{cases} \tag{7}$$

where $\delta \in (0, 1]$ *is a fixed parameter.*

Assumption 2 states that retaining a higher ownership share of the intangible, θ, requires the entrepreneur to choose a smaller span. Indeed, $\partial_\theta f(\theta, \delta) = -\frac{1}{\delta\theta} < 0$. The strength of this effect is governed by the parameter δ. When δ is close to zero, the entrepreneur can increase span without giving up a large portion of her endowment of intangibles. When δ is large, on the other hand, increasing span requires forfeiting more intangibles. Thus, δ captures how easy it is for the entrepreneur to retain the surplus generated by the intangible capital as the firm grows.

Under this assumption, the entrepreneur jointly chooses span and the share of the intangible endowment to retain, θ, as follows:

$$\hat{V}_e(N) = \max_{\theta \in [0,1],\ x \geq 0} \theta V(N, x) \quad \text{s.t.} \quad x = f(\theta, \delta). \tag{8}$$

The solution is:

$$\hat{x} = \frac{\rho}{\delta}, \quad \hat{\theta} = \delta e^{-\rho},$$

and

$$\hat{V}_e(N) = A\, N\, \delta e^{-\rho} \left(\frac{\rho}{\delta}\right)^{\rho}.$$

A high degree of non-rivalry (ρ close to 1) is associated with high firm span (high $\hat{x}$) but low retention of intangibles by the entrepreneur (low $\hat{\theta}$). A low degree of non-rivarly (ρ close to 0), or high costs of storing the intangible externally (δ high) is associated with low firm span (low $\hat{x}$) but high retention of intangibles by the entrepreneur (high $\hat{\theta}$).

We note that these comparative statics of the solution to the entrepreneur's problem hold outside of the specific functional form for $f(\theta, \delta)$, the function linking span and retention, described in Equation 7. Appendix A.1 provides sufficient conditions on the function $f(\theta, \delta)$ under which the degree of non-rivalry, ρ, is positively related to optimal span, $\hat{x}$, and negatively related to optimal retention, $\hat{\theta}$, while limits to excludability are positively related to retention. The key condition is that the (absolute value of) the elasticity of f with respect to θ be declining with δ. If that is the case, then as δ increases, a marginal increase in firm span requires the entrepreneur to give more of the intangible asset to outsiders. As a result, the entrepreneur tends to limit span more (lower $\hat{x}$), and keep the firm more closely held (high $\hat{\theta}$).

Assumption 2 captures the idea that appropriating the returns that intangibles generate may be difficult for the entrepreneur. But Assumption 2 goes beyond this idea by specifying that these frictions are exacerbated by x, the span of the firm. The parameter δ indexes how important these frictions are. We propose two main interpretations of this assumption:

Interpretation 1: imitation As the firm grows and its span expands, its intangibles are used across more product streams, which could capture markets in different locations. The broader dissemination of the firm's intangibles may make it easier for other firms to replicate and use the intangibles themselves. Imitation by other firms may in turn lower the effective return to the intangible for the firm that created it. If imitation is more likely as the firm expands across more markets, this creates a force that offsets the benefits from non-rivalry.

An example of this process is competition by latecomers in app-based services, like ride-hailing services, food-delivery services, or bike share services. The creation of an intangible asset (a ride-hailing app) by a first entrant (Uber) might soon be imitated by latecomers (Lyft), eroding the returns to the creation of the intangible for the first entrant.

Under this interpretation, the parameter δ captures limits to excludability arising from imitation, with higher values of δ corresponding to larger costs associated with imitation. These costs reflect both the nature of the storage technology (intangibles which are easy to store, such as algorithms stored in software, might also be easier to imitate), and the property rights institutions that protects ownership of the intangible, as a lack enforceability of property rights by the judicial system may encourage imitation.

Interpretation 2: incomplete contracts As the firm grows and its span expands, it may require financing from outsiders. However, if the firm's intangibles remain closely held by firm insiders (such as entrepreneurs or key employees), investment by outsiders may be deterred by hold-up or asymmetric information problems. A solution is for insiders to codify and externally store the intangible, so that cash flow or control rights can be assigned to outsiders. Once externally stored, returns to the firm's intangibles can credibly be pledged, making external financing possible.

An example of this process is patent-filing for firms in an early stage R&D program, such as biotech startups, that requires financial backing from outsiders. Patent-filing has been shown to improve access to capital and growth for early-stage startups (Graham, Merges, Samuelson, and Sichelman, 2009; Helmers and Rogers, 2011). Patent-filing is a way to codify the underlying intangible and signal its value to outsiders, as well as to ensure that it has salvage value, making pledgeability and external financing possible.

Under this interpretation, the parameter δ captures the costs associated with the external financing process, with higher values corresponding to more costly external financing. These costs will tend to be higher when the intangible storage technology is difficult to replicate (for instance, intangible stored in key talent, or organizational capital), if future returns to the intangible are difficult to evaluate (for instance, patents for breakthrough innovations), or when the institutional environment makes it difficult to recognize and assign property rights to intangible assets. In those cases, insiders will need to pledge more of the intangibles in order to raise external financing, limiting the potential span of the firm.

Under both interpretations, the choice of scale by the entrepreneur is inefficient. This follows from the fact that the entrepreneur chooses x to maximize (8), rather than $V(N, x)$. The latter value function, which corresponds to either the social value of the intangible (interpretation 1) or the enterprise value of the firm (interpretation 2), can be written (given the optimal choice of the entrepreneur $\hat{x}$) as

$$\hat{V}(N) = A\,N\,\hat{x}^{\rho} = A\,N\,\left(\frac{\rho}{\delta}\right)^{\rho}.$$

Because of limited excludability, the entrepreneur always chooses a smaller span than the span that maximizes enterprise value ($x = +\infty$).

To obtain some further intuition about the interaction of non-rivalry with limited excludability, Figure 2 illustrates the comparative statics of the model with respect to ρ, non-rivalry, and δ, the limits to excludability.

The optimal span $\hat{x} = \rho/\delta$ increases with the degree of non-rivalry. However, for given limits to excludability, δ, a higher degree of non-rivalry does not necessarily make the entrepreneur better off. The left panel of Figure 2 illustrates this. When excludability is high (δ is low), given the option to adopt an intangible-intensive technology with a high non-rivalry (ρ to 1), versus using only rival capital inputs ($\rho = 0$), the entrepreneur would generally pick the former, and operate at high scale. However, when excludability is low (δ is high), that is not the case: the entrepreneur might instead pick a technology with rival capital inputs, $\rho = 0$, and focus on a single production stream ($\hat{x} = 0$).

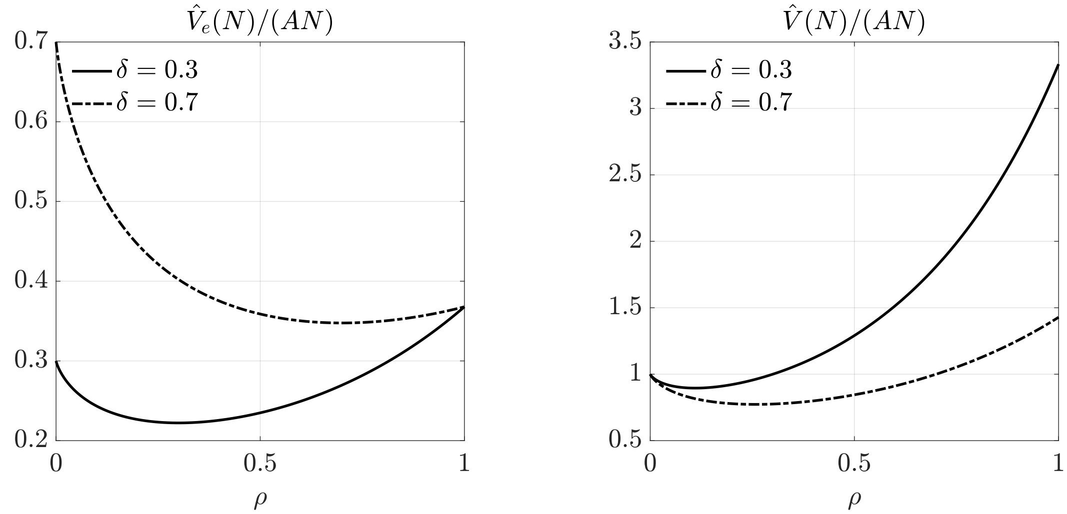

Figure 2: Comparative statics with respect to ρ and δ. The left panel reports the entrepreneur's value function, $\hat{V}_e(N)$, normalized by AB, where $A = (1-\zeta)(\zeta/R)^{\frac{\zeta}{1-\zeta}}$ and N is the intangible stock. The right panel report total enterprise value $\hat{V}(N)$, normalized by AN.

This property demonstrates the complementarity between non-rivalry and excludability, which we also emphasize in Crouzet et al. (2022). Non-rivalry may increase the returns to intangibles, but the entrepreneur will only value the associated intangible asset to the extent that the benefits are appropriable. To see this, note that the cross-partial derivative of the (log of) entrepreneur's value function is equal to:

$$\partial_{\rho\delta} \log(\hat{V}_e(N)) = -1/\delta < 0,$$

so that non-rivalry (high ρ) and excludability (low δ) are complements.

Finally, the comparison of the left and right panels of Figure 2 further illustrate the fact that the entrepreneur's scale choices may be inefficient. In the right panel, for all values of δ, a non-rival technology (ρ close to 1) yields higher social value (under the interpretation of δ relating to positive spillovers to other firms) or enterprise value (under the interpretation of δ as relating to imperfect pledgeability). This conflicts with the preferences of the entrepreneur, who would rather choose a technology with non-scalable inputs when δ is sufficiently high.

Stage 1: Intangible investment

The last step is to determine the initial amount of the intangible asset N. To do so, we need to take the perspective of the entrepreneur who invests in producing new intangibles. More specifically, the entrepreneur exerts effort ι subject to a convex cost $c(\iota)$ to generate a new intangible asset. The process of generating new intangibles can be risky: exerting effort ι yields intangible capital $N \sim f(N;\iota)$. Exerting higher effort yields ex-ante better outcomes: we assume that if $\iota' > \iota$ then

$f(N;\iota') \overset{\text{fosd}}{>} f(N;\iota)$. Given these assumptions, the entrepreneur solves

$$\max_{\iota} \int \hat{V}_e(N) f(N;\iota) dN - c(\iota) \tag{9}$$

which after substituting for (9) yields the optimality condition:

$$A\left[\delta e^{-\rho}\left(\frac{\rho}{\delta}\right)^{\rho}\right]\frac{\partial}{\partial e}E[N;\hat{\iota}] = \frac{\partial}{\partial \iota}c(\hat{\iota}). \tag{10}$$

Examining (10), we see that the dependence of the entrepreneur's optimal effort choice on ρ and δ is determined by how the term in brackets depends on ρ and δ.

We can immediately see that $\partial\hat{\iota}/\partial\delta < 0$; if the entrepreneur needs to give up more value in order to scale her firm (δ is higher) then her marginal valuation of the intangible N is lower, which leads to lower ex-ante investment in generating intangibles. By contrast, the comparative statics with respect to the non-rivalry parameter ρ are more subtle. It turns out that the term in brackets is increasing in ρ if $\rho > \delta$, and decreasing otherwise. This is again related to the complementarity emphasized earlier between non-rivalry and excludability. Non-rivalry may generate value, but the entrepreneur will only value the associated intangible asset to the extent that the benefits are sufficiently appropriable. Otherwise, if the entrepreneur cannot appropriate a significant share of the rents (δ is high enough), she will exert less effort in generating new intangibles for local increases in non-rivalry ρ.

Additionally, the model features underinvestment in innovation since the entrepreneur's effort choice depends on her private value of the intangible, which in general is lower than the social value. This is similar to models of endogenous growth with spillovers. Perhaps surprisingly, however, we see that the degree of under-investment can be greater for intangibles that are highly scalable (higher ρ) if appropriability is low enough (δ is high).

Last, it is important to emphasize the distinction between ex-post rents to the entrepreneur

$$A\left[\delta e^{-\rho}\left(\frac{\rho}{\delta}\right)^{\rho}\right]N - c(\hat{\iota}), \tag{11}$$

with ex-ante rents

$$A\left[\delta e^{-\rho}\left(\frac{\rho}{\delta}\right)^{\rho}\right]E[N;\hat{\iota}] - c(\hat{\iota}). \tag{12}$$

If there is selection on which entrepreneurs enter the market (or equivalently if failure $N = 0$ is a feasible outcome despite the amount of effort involved) then focusing on ex-post compensation to entrepreneurs (11) will overstate their payoff. Put simply, ex-post rents (11) can be positive even if rents are zero ex-ante (12) due to free entry of entrepreneurs.

2.2 Comparison to existing approaches

We conclude by discussing the relationship between our model, and the way in which the key properties of intangibles have been discussed in existing work.

Scalability

A common assumption in the literature is that intangible capital can contribute to "higher returns to scale", or more generally, that intangibles are "scalable" (Haskel and Westlake, 2017). This property is often thought to be connected to the non-rival nature of intangible capital as a production input. To explain this, a distinction is usually made between the cost of *creating* new intangible assets, and the cost of *using* existing intangible assets.

Like tangible assets, creating new intangible assets is costly, in the sense that it requires investment on the part of the firm together with inputs that may be in scarce supply. However, once a new intangible asset is created, the cost of *using* it repeatedly is quite low or even zero. The low usage cost within the firm is analogous to a high value of ρ in the model. The fact an intangible asset is being used to produce one unit of a good or service for a customer does not fully preclude it from being used to produce the same good or service for another customer.

In existing work, this is often captured by assuming that intangibles involves high fixed costs, but leads to lower marginal cost of production for the firm (Hsieh and Rossi-Hansberg, 2019; Kwon, Ma, and Zimmermann, 2021). As a result, production at intangible-intensive firms may be characterized by increasing returns to scale, at least locally.

Instead of assuming a particular cost structure with increasing returns, our model starts from a formal definition of non-rivarly in use, Assumption 1. Interestingly, the notion of "scalability" that results from this approach is somewhat different from the existing literature. Namely, "scalability" in our model is best understood as the fact intangible capital and firm span are *complements*. To see why, recall that firm value at Stage 3 is given by $V(N, x) = A\, N\, x^{\rho}$. When $\rho = 0$, we have:

$$\frac{\partial V(N,x)}{\partial x} = 0, \quad \frac{\partial V(N,x)}{\partial N} = A.$$

In this case, the marginal value of intangible capital is independent of the span of the firm x, and thus intangibles do not benefit from greater scale. Similarly, increasing the amount of intangibles deployed in the firm does not affect the benefit of increasing scale x. By contrast, when $\rho \to 1$,

$$\frac{\partial V(N,x)}{\partial x} = A\, N, \quad \frac{\partial V(N,x)}{\partial N} = A\, x.$$

In this case, firm scale and the stock of intangibles become complements: the value of the intangible asset increases when it can be employed in multiple segments. Similarly, the higher the value of the firm's intangible asset, the greater is the benefit of expanding the span x of operations.

Thus, from the assumption of non-rivalry in use, our model derives a novel characterization of "scalability" or "returns to scale" for intangibles, namely that they act as complements with firm span – and more so, the stronger is non-rivalry (i.e., the higher is ρ).

Limits to appropriability

The fact that intangibles can be stored externally and copied, combined with imperfections in the enforcement of exclusion, imply that the returns generated by a particular intangible asset may be difficult to fully appropriate by the firm or agent that created it. For instance, competitors can imitate a firm's process by copying their key patent, algorithm, or software. Because of the non-rivalry of intangibles, this process is substantially simpler than for replicating, say, the competitor's production facilities. Additionally, this imitation process is facilitated by environments with weak intellectual property rights protection. Eventually, this replication can erode the returns generated by the intangible for its original creator. In our model, this process is captured by the assumption that $\delta > 0$, so that the entrepreneur can only appropriate the returns generated by a portion of the intangibles she initially owns.

We note that the flipside of limits to appropriability is that intangibles generate spillovers outside of the firm. Ideas that are stored and widely disseminated can be used effectively in production — or even spur the development of better ideas. As a result, the use of a specific intangible asset by one firm may indirectly increase productivity in other firms who can potentially adopt the same intangible. This notion of positive spillovers is present in many models of endogenous growth in the tradition of Romer (1990). Though positive spillovers are not fully fleshed out in our model, we note that it is somewhat captured by the fact that the overall social or welfare value of the project, $\hat{V}(N)$, can exceed the value that is privately appropriable by the entrepreneur, $\hat{V}_e(N)$.

Here, we should emphasize that negative spillovers are also possible. The same forces that lead to wide dissemination and adoption of new ideas imply that older ideas become more easily obsolete. A new and more efficient method of production can be licensed to many firms, leading to a drop in the value of the intangible asset representing the old production method (e.g. the patent). The assumption that $\delta > 0$ could also capture this process of "external" depreciation of the firm's assets.

3 Economic implications

In this section, we discuss the implications of the model for four issues in macroeconomics and finance: productivity growth; factor income shares; Tobin's Q and investment; market structure. Throughout this section, we focus on the case when the function relating firm span and the fraction of intangibles retained by the entrepreneur is:

$$x = f(\theta, \delta) = f(\theta, \delta) = \begin{cases} -\frac{1}{\delta} \log\left(\frac{\theta}{\delta}\right) & \text{if} \quad \theta \in [0, \delta) \\ 0 & \text{if} \quad \theta \in [\delta, 1] \end{cases}.$$

3.1 Aggregate Productivity

Consider the model in the previous section. If we assume that there is no heterogeneity in ρ and δ and then clear the market for physical capital to determine the equilibrium interest rate R, we can write aggregate output Y with some abuse of notation as

$$Y = \left(\frac{\rho}{\delta}\right)^{\rho\,(1-\zeta)} N^{1-\zeta}\, K^{\zeta}. \tag{13}$$

As we examine (13), it is useful to keep in mind that the simple model from the previous section has constant total factor productivity. As such, aggregate output is a function of the economies' stock of physical capital K, the quantity of intangibles N, with an adjustment for the fact that intangibles are non-rival (the first term). Here, for simplicity, we ignore the effort cost e to generate new intangibles when constructing output (consistent with the data). Consider the extreme case where the only measurable input to production is physical capital, that is, aggregate statistics can only measure Y_t and K_t. Taking logs of (13), we can define measured TFP (in logs) as

$$tfp \equiv \log Y - \zeta \log K = \rho\,(1-\zeta)\;(\log\rho - \log\delta) + (1-\zeta)\log N. \tag{14}$$

Examining equation (14), we see that measured productivity depends not only on the 'stock' of intangible capital N but also on their degree of non-rivalry and appropriability (i.e. ρ and δ). Specifically, the non-rivalrous nature of intangibles can imply that once an intangible asset is developed, output can increase rapidly as the intangible capital is applied in many locations or applications simultaneously (which depends on ρ). Similarly, the optimal scale of deployment is a function of appropriability (determined by δ).

Equation (14) can then be interpreted from two perspectives: either intangibles account for the entirety of the Solow residual; or they pose a measurement challenge for which Solow residual measures must adjust. Crouzet et al. (2022) discuss these two perspectives in more detail.

3.2 Factor Income Shares

The fact that intangibles are typically hard to measure implies that factor shares are also mismeasured. Depending on the implicit assumptions researchers make, the share of output that would accrue to intangibles is can be allocated to either physical capital, labor, or 'rents', where the latter is defined as monopoly profits. As an illustration, let us now re-interpret the fixed factor K in the model as a composite input good consisting of physical capital M and labor L, both in fixed supply,

$$K \equiv M^{\alpha} L^{1-\alpha}. \tag{15}$$

In this case, the share of output that accrues to 'intangibles' (i.e. not to K) is equal to

$$\frac{Y - R\,K}{Y} = 1 - \zeta, \tag{16}$$

while the factor share of physical capital and labor is $\alpha\,\zeta$ and $(1-\alpha)\,\zeta$, respectively.

Now, the question is whether the factor share of intangibles (16) should be allocated to labor or capital. If we view δ as reflecting frictions between the entrepreneur and outside financiers, then we can

$$\frac{N_e}{N} = \delta e^{-\rho} \qquad \text{and} \qquad \frac{N - N_e}{N} = 1 - \delta e^{-\rho} \tag{17}$$

as the share of intangibles that accrues to the entrepreneur, and to outside investors, respectively. The greater the degree of non-rivalry (higher ρ) the smaller is the share that accrues to the entrepreneur, as she chooses to give up a larger fraction of her surplus to achieve a higher scale. The stronger the limits to excludability (higher δ), the more of the surplus the entrepreneur chooses to retain — as limited pleadgeability make expansion too costly for her.

Equation (17) gives some guidance on how the factor shares of intangibles should be allocated between capital and labor. The entrepreneur's share N_e/N should likely be considered labor income if it is the case that human capital is the key input in the production of new intangibles. The residual part $1 - N_e/N$, however, is the part that outside investors have a claim to, which could (though it need not be) be part of capital income.

3.3 Physical Investment and Tobin's Q

Recent work has highlighted the fact that while investment rates in physical capital have generally trended downward over the last two decades, proxies for the incentive to invest, such as Tobin's average Q, have shown no such trend (Gutiérrez and Philippon, 2016; Crouzet and Eberly, 2021). From the standpoint of neoclassical investment models, these trends can be viewed as indicating that the wedge between average Q (the empirical proxy for the incentive to invest) and marginal q (the actual marginal return to physical investment) has increased. The model outlined in the previous section can shed some light on the role that intangibles may have played in this process.

In order to make the connection to Q-theory clearer, we assume that the total stock of physical capital is fixed, instead of rented on markets at a fixed marginal cost R. A firm with installed physical capital K, intangibles N, and span x has value:

$$
\begin{aligned}
W(K,N,x) \quad &= \quad \max_{\{N(s),K(s)\}_{s\in[0,x]}} \quad \int_0^x N(s)^{1-\zeta} K(s)^{\zeta} ds && (18)\\
\text{s.t.} \quad [q_K] \quad & \int_0^x K(s)\, ds \le K && (19)\\
[q_N] \quad & \left(\int_0^x N(s)^{\frac{1}{1-\rho}}\, ds\right)^{1-\rho} \le N, && (20)
\end{aligned}
$$

where q_K and q_N are the Lagrange multipliers on the physical and intangible capital allocation constraints, that is, the marginal increase in firm value associated with a marginal increase in K or in N:

$$
\frac{\partial W}{\partial K} = q_K, \quad \frac{\partial W}{\partial N} = q_N.
$$

Following similar steps as before, the solution for enterprise value and these shadow values is:

$$
\begin{aligned}
W(K,N,x) \quad &= \quad N^{1-\zeta} K^{\zeta} x^{\rho(1-\zeta)}, && (21)\\
q_K \quad &= \quad \zeta \left(\frac{N}{K}\right)^{1-\zeta} x^{\rho(1-\zeta)}, && (22)\\
q_N \quad &= \quad (1-\zeta) \left(\frac{K}{N}\right)^{\zeta} x^{\rho(1-\zeta)}. && (23)
\end{aligned}
$$

Note that, if we impose $q_K = R$, we obtain:

$$
K = \left(\frac{\zeta}{R}\right)^{\frac{1}{1-\zeta}} N x^{\rho} \quad \text{and} \quad V(N,x) = W(K,N,x) - RK = (1-\zeta)\left(\frac{\zeta}{R}\right)^{\frac{\zeta}{1-\zeta}} N x^{\rho},
$$

that is, the same solution as in the model of the previous section. Thus the model of the previous section can be thought of as a particular case of the fixed-capital model, if q_K is set fixed to $q_K = R$.

The solution in Equations (21)-(23) implies the following firm value decomposition:

$$
W(K,N,x) = q_K K + q_N N. \tag{24}
$$

This decomposition offers some insight into the growing disconnect between average Q_K and marginal

q_K for physical capital. Rewriting Equation (24) as:

$$\frac{W}{K} \equiv Q_K = q_K + q_N \frac{N}{K}, \tag{25}$$

we see that intangibles introduce a wedge between average Q_K and marginal q_K. The wedge, $Q_K - q_K = \frac{N}{K} q_N$, depends positively on the ratio of intangible to physical capital, N/K, since:

$$Q_K - q_K = q_N \frac{N}{K} = (1-\zeta) \left(\frac{N}{K}\right)^{1-\zeta} \left(\frac{\rho}{\delta}\right)^{\rho(1-\zeta)}, \tag{26}$$

where we used the fact that the optimal choice of span in the model of the previous section is $\hat{x} = \frac{\rho}{\delta}$.

In a dynamic extension of this model, physical investment rates would be an increasing function of the marginal value of physical capital q_K. Thus, the growing disconnect between average Q_K and physical investment rates (a function of marginal q_k could be explained by an rising ratio of intangible to physical capital N/K, as in Hayashi and Inoue 1991 and Crouzet and Eberly 2018.

However, for a given ratio of intangible to physical capital, the model has an additional prediction: it indicates that the wedge depends positively on the ratio $\left(\frac{\rho}{\delta}\right)^{\rho}$. In particular, if non-rivalry is stronger (ρ is higher), then the total contribution of intangibles to firm value is higher, magnifying the wedge between average and marginal q for physical assets. Likewise, if excludability is weaker (δ is higher), the contribution of intangible to firm value is lower, reducing the wedge between average and marginal q.

3.4 Rents

An alternative explanation for the growing disconnect between average Q and marginal q is growing market power, in either the product or input. Growing market power implies that firms have an incentive to reduce scale, and therefore invest less. This mechanism is forcefully illustrated in Gutiérrez and Philippon (2016) and Gutierrez and Philippon (2019).

Market power and rents can be introduced in the model of the previous section, and interact with the specific properties of intangibles (non-rivarly and limited excludability) in rich ways. Consider the fixed-capital model described in Equations (18)-(20), but assume that total sales are given by:

$$\left(\int_0^x N(s)^{1-\zeta} K(s)^{\zeta}\right)^{\frac{1}{\mu}},$$

where $\mu > 1$ is a fixed parameter that creates a wedge between the average and marginal revenue product of both K and N. The wedge is a simple way to capture the rents associated with production, and could be microfounded, for instance, as a markup of output prices over marginal cost in a monopolistic setting.[3] In this case, firm value W, and marginal q_K and q_N are given by:

$$W(K,N,x) = x^{\frac{(1-\zeta)\rho}{\mu}} N^{\frac{1-\zeta}{\mu}} K^{\frac{\zeta}{\mu}} \tag{27}$$

$$q_K = \frac{\zeta}{\mu}\left(\frac{N}{K}\right)^{1-\zeta}\left(\frac{K^\zeta}{N^{1-\zeta}}\right)^{1-\frac{1}{\mu}} x^{\frac{\rho(1-\zeta)}{\mu}} \tag{28}$$

$$q_N = \frac{1-\zeta}{\mu}\left(\frac{K}{N}\right)^{\zeta}\left(\frac{K^\zeta}{N^{1-\zeta}}\right)^{1-\frac{1}{\mu}} x^{\frac{\rho(1-\zeta)}{\mu}} \tag{29}$$

This solution implies a more general version of the firm value decomposition in Equation (24):

$$W(K,N,x) = \overbrace{q_K K + q_N N}^{(a)} + \underbrace{(\mu-1)q_K K + (\mu-1)q_N}_{(b)}. \tag{30}$$

This decomposition highlights two broad sources of firm value. Term (a) is the value of installed capital, which makes up all of firm value in the absence of rents ($\mu = 1$), as was the case in Equation (24). Term (b) is the net value of rents, which is positive only when $\mu > 1$. Additionally, each of these two terms (value of installed assets, and rents) can be decomposed between a contribution of physical and a contribution of intangible capital. Consistent with the idea that the intangible asset itself is not the rent, this decomposition assumes no specific relationship between μ (a fixed parameter) and the stock of intangibles (N, which we take as given in this application of the model).

To see the effect of non-rivalry and limited excludability on rents, define total rents per unit of physical capital:

$$\Gamma(K,N,x) \equiv (\mu-1)\left(q_K + q_N\frac{N}{K}\right), \tag{31}$$

and it is straightforward to see that:

$$\Gamma(K,N,x) = \frac{\mu-1}{\mu}\left(\frac{N}{K}\right)^{1-\zeta}\left(\frac{K^\zeta}{N^{1-\zeta}}\right)^{1-\frac{1}{\mu}} x^{\frac{\rho(1-\zeta)}{\mu}}. \tag{32}$$

Of these rents, we assume that only $\Gamma^e(K,N,x)$ are appropriable by the entrepreneur, where:

$$\Gamma^e(K,N,x) \equiv \delta e^{-\frac{\delta(1-\zeta)x}{\mu}}\Gamma(K,N,x). \tag{33}$$

[3] We choose to introduce this wedge to the sum of all production across streams. If, instead, it applied individually to each stream, i.e. $Y(s) = \left(N(s)^{1-\zeta}K(s)^\zeta\right)^{1/\mu}$, then the firm would have a motive to increase its span even when $\rho = 0$, because a higher span would counterbalance decreasing returns at the stream level. We leave this mechanism out of the model, since it is independent of the non-rivalry of intangibles, but it is possible to include it in the model. It generally leads to a higher choice of span, all other things equal.

Note that here, for simplicity, we assumed that span x and retention θ are related through:

$$x(\theta) = -\frac{1-\zeta}{\mu\delta} \log\left(\frac{\theta}{\delta}\right),$$

which helps clarify the parallels between this model and the model with variable capital and no rents. Indeed, as before, we assume that the entrepreneur chooses the span x to maximize her claim, which, in this model, is proportional to the value of rents she can appropriate. The optimal span is the same as before, namely $\hat{x} = \frac{\rho}{\delta}$, implying:

$$\hat{\Gamma}(K,N) = \frac{\mu-1}{\mu}\left(\frac{N}{K}\right)^{1-\zeta}\left(\frac{K^{\zeta}}{N^{1-\zeta}}\right)^{1-\frac{1}{\mu}}\left(\frac{\rho}{\delta}\right)^{\frac{(1-\zeta)\rho}{\mu}}, \tag{34}$$

$$\hat{\Gamma}^{e}(K,N) = \delta e^{-\frac{(1-\zeta)\rho}{\mu}}\hat{\Gamma}(K,N). \tag{35}$$

Relative to a model with $\rho = 0$ and $\delta = 0$, this expression shows that both limits to excludability and non-rivalry affect both the total size of rents. Higher non-rivalry (ρ closer to 1) generally leads to a lower share of rents accruing to the entrepreneur, while lower excludability (higher δ) generally implies that the entrepreneur retains a higher share of total rents.

4 Conclusion

We propose a simple model that captures the two core economic properties of intangibles arising from the fact they consist of information in need of a storage medium. First, to the extent that the storage medium can easily be replicated, intangibles exhibit some degree of *non-rivalry* in use (the same intangible can be used simultaneously in multiple production streams. Non-rivalry implies that intangibles are scalable: the stock of intangibles of the firm, and the firm's span or scope, are complements. Second, to the extent that the intangible is easy to copy, it can also be appropriated by outsiders implying limits to excludability. Thus, imperfect excludability can also limit the incentive for entrepreneurs, managers, or key personnel to create and develop intangibles, potentially leading to inefficiently low investment. That is, even once the intangible is stored in a particular medium, it can be difficult to assign control and cash flow rights to the surplus that it creates. Without some degree of excludability, intangible capital cannot become an intangible asset. Last, we discuss how both non-rivalry and limits to excludability can contribute to understanding four recent macroeconomic and financial trends: the decline in measured aggregate productivity; the rise in the labor income share; the growing divergence between Tobin's Q and investment; and the rise in rents.

References

Chandler, A. (1993). The Visible Hand: The Managerial Revolution in American Business. Harvard University Press.

Crouzet, N. and J. Eberly (2018). Intangibles, investment, and efficiency. American Economic Review, Papers and Proceedings 108, 426–31.

Crouzet, N., J. Eberly, A. Eisfeldt, and D. Papanikolaou (2022). The economics of intangible capital. The Journal of Economic Perspectives forthcoming.

Crouzet, N. and J. C. Eberly (2021, July). Rents and Intangible Capital: A Q+ Framework. Working Paper 28988, National Bureau of Economic Research.

Eisfeldt, A. L. and D. Papanikolaou (2014). The value and ownership of intangible capital. American Economic Review 104(5), 189–94.

Graham, S. J., R. P. Merges, P. Samuelson, and T. Sichelman (2009). High technology entrepreneurs and the patent system: Results of the 2008 berkeley patent survey. Berkeley Technology Law Journal, 1255–1327.

Gutiérrez, G. and T. Philippon (2016). Investment-less growth: An empirical investigation. Technical report, National Bureau of Economic Research.

Gutierrez, G. and T. Philippon (2019). The failure of free entry. Working Paper 26001, National Bureau of Economic Research.

Hall, R. E. (2001). The stock market and capital accumulation. American Economic Review 91(5), 1185–1202.

Haskel, J. and S. Westlake (2017). Capitalism without capital. Princeton University Press.

Hayashi, F. and T. Inoue (1991). The relation between firm growth and q with multiple capital goods: Theory and evidence from panel data on japanese firms. Econometrica: Journal of the Econometric Society, 731–753.

Helmers, C. and M. Rogers (2011). Does patenting help high-tech start-ups? Research Policy 40(7), 1016–1027.

Hsieh, C.-T. and E. Rossi-Hansberg (2019). The industrial revolution in services. Technical report, National Bureau of Economic Research.

Kwon, S. Y., Y. Ma, and K. Zimmermann (2021). 100 years of rising corporate concentration. Available at SSRN 3936799.

McGrattan, E. R. and E. C. Prescott (2010). Technology capital and the us current account. 100(4), 1493–1522.

Romer, P. M. (1990). Endogenous technological change. Journal of political Economy 98(5, Part 2), S71–S102.

A Appendix to Section 2

In this section, we provide additional results for the model of Section 2.

A.1 The link between span and retention

The results of Section 2 can also be established for more general functional forms linking the span of the firm and the degree of retention by the entrepreneur. We start by giving some general sufficient conditions under which the comparative statics of the solution are similar to those obtained in the main text.

Proposition 1. *Assume that* $f \in \mathcal{C}^2\left([0,1]^2, \mathbb{R}_+\right)$ *satisfies* $f(\theta,\delta) > 0$ *for all* $\theta, \delta \in (0,1)^2$. *Define:*

$$\eta(\theta,\delta) \equiv -\frac{\theta \partial_\theta f(\theta,\delta)}{f(\theta,\delta)}.$$

If η *is positive and increasing with* θ *on* $[0,1]$ *and satisfies* $0 < \lim_{\theta\to 0^+} \eta(0,\delta) \leq 1$ *and* $\lim_{\theta \to 1^-} \eta(1,\delta) = +\infty$*, then there is a unique interior solution to the entrepreneur's problem in Stage 2, characterized by:*

$$\eta(\hat{\theta}(\rho,\delta),\delta) = \frac{1}{\rho}, \quad \hat{x}(\rho,\delta) = f(\hat{\theta}(\rho,\delta),\delta). \tag{36}$$

The solution satisfies:

$$\partial_\rho \hat{\theta} > 0, \quad \partial_\rho \hat{x} < 0.$$

Moreover, if:

$$\partial_\theta \eta(\theta,\delta) \leq 0, \tag{37}$$

then the solution satisfies:

$$\partial_\delta \hat{\theta} \geq 0.$$

Proof. The objective function of the entrepreneur is proportional to θx^θ. Replacing $x = f(\theta,\delta)$, the first derivative of the objective function has the same sign as:

$$1 - \rho\eta(\theta,\delta). \tag{38}$$

The assumptions in Equation 36 ensure that for any $(\rho,\delta) \in [0,1]^2$, there is a unique $\hat{\theta}$ such that:

$$1 - \rho\eta(\hat{\theta},\delta) = 0,$$

and moreover, that $\hat{\theta}$ maximum of the objective function. Moreover, we have:

$$\begin{aligned} \partial_\rho \hat{\theta} &= -\frac{\eta}{\rho \partial_\theta \eta} < 0, \\ \partial_\rho \hat{x} &= (\partial_\rho \hat{\theta})(\partial_\theta f) > 0, \end{aligned}$$

where we used the assumption that $\partial_\theta \eta < 0$. Finally, note that:

$$\partial_\delta \hat{\theta} = -\frac{\partial_\delta \eta}{\partial_\theta \eta} \geq 0.$$

□

Next, we provide examples of functions that satisfy conditions (36) and (37).

Example 1. *Let $\delta \in (0,1]$. For the function:*

$$f(\theta,\delta) = (-\log(\theta))^{\frac{1}{\delta}}, \tag{39}$$

the elasticity η is given by:

$$\eta(\theta,\delta) = \frac{-1}{\delta \log(\theta)}.$$

This elasticity is increasing in θ, satisfies:

$$\lim_{\theta\to 0^+} \eta(\theta,\delta) = 0, \quad \lim_{\theta\to 1^-} \eta(\theta,\delta) = +\infty.$$

Therefore, the unique solution to the entrepreneur's problem in Stage 2 is given by:

$$\hat{\theta} = e^{-\frac{\rho}{\delta}}, \quad \hat{x} = \left(\frac{\rho}{\delta}\right)^{\frac{1}{\delta}}.$$

Moreover, $\hat{\theta}$ is strictly decreasing with ρ, $\hat{x}$ is strictly increasing with ρ, and, since,

$$\partial_\delta \eta = \frac{1}{\delta^2 \log(\theta)} < 0,$$

$\hat{\theta}(\rho,\delta)$ is strictly increasing in δ.

In this example, note we can write the relation between θ and x as:

$$\theta(x,\delta) = \exp(-x^\delta).$$

This function satisfies:

$$\partial_\delta \theta(x,\delta) = -\log(x) x^\delta \exp(-x^\delta) \gtrless 0.$$

Therefore, in order to achieve a given desired firm span, x, a higher value of δ need not require the entrepreneur to retain less of the intangible.

Example 2. *Let $\delta \in (0,1]$ and $x_m > 0$. For the function:*

$$f(\theta,\delta) = x_m(1-\theta)^{\frac{1}{\delta}}, \tag{40}$$

the elasticity η is given by:

$$\eta(\theta,\delta) = \frac{1}{\delta}\left(\frac{1}{1-\theta} - 1\right).$$

This elasticity is increasing in θ, satisfies:

$$\lim_{\theta\to 0^+} \eta(\theta,\delta) = 0, \quad \lim_{\theta\to 1^-} \eta(\theta,\delta) = +\infty.$$

Therefore, the unique solution to the entrepreneur's problem in Stage 2 is given by:

$$\hat{\theta} = \frac{\delta}{\rho+\delta}, \quad \hat{x} = x_m\left(\frac{\rho}{\rho+\delta}\right)^{\frac{1}{\delta}}.$$

Moreover, $\hat{\theta}$ is strictly decreasing with ρ, $\hat{x}$ is strictly increasing with ρ, and, since,

$$\partial_\delta \eta = -\frac{1}{\delta^2}\left(\frac{1}{1-\theta} - 1\right) < 0,$$

$\hat{\theta}(\rho, \delta)$ is strictly increasing in δ.

In this example, the parameter x_m captures the maximum span which an entrepreneur could choose, and which would maximize the overall value of the firm. As long as $\delta > 0$, though, limited excludability will reduce firm scope below that maximum.

Example 3. *Let $\delta \in (\rho, 1]$. For the function:*

$$f(\theta, \delta) = \left(\frac{\theta}{1-\theta}\right)^{-\frac{1}{\delta}}, \tag{41}$$

the elasticity η is given by:

$$\eta(\theta, \delta) = \frac{1}{\delta(1-\theta)}$$

This elasticity is increasing in θ, satisfies:

$$\lim_{\theta \to 0^+} \eta(\theta, \delta) = \frac{1}{\delta} < \frac{1}{\rho}, \quad \lim_{\theta \to 1^-} \eta(\theta, \delta) = +\infty.$$

Therefore, the unique solution to the entrepreneur's problem in Stage 2 is given by:

$$\hat{\theta} = 1 - \frac{\rho}{\delta}, \quad \hat{x} = \left(\frac{\delta}{\rho} - 1\right)^{-\frac{1}{\delta}}.$$

Moreover, $\hat{\theta}$ is strictly decreasing with ρ, $\hat{x}$ is strictly increasing with ρ, and, since,

$$\partial_\delta \eta = -\frac{1}{\delta^2(1-\theta)} < 0,$$

$\hat{\theta}(\rho, \delta)$ is strictly increasing in δ.

In this example, the interior first-order condition only characterizes a global optimum when δ is sufficiently high; otherwise, the optimal retention is $\theta = 0$, and the optimal scale is infinite. Additionally, the function $f(\theta, \delta)$ in this example is connected to the log-logistic distribution. Indeed, it satisfies:

$$f(\theta, \delta) = F^{-1}(1 - \theta; \delta),$$

where $F(,;\delta)$ is the CDF of the log-logistic distribution with shape parameter δ. More generally, examples of functions linking span and retention can be built from the tail probability of CDFs supported on the positive real line.s